PBS Mathline®

Rhinos and M&M's®

(Exponential Models)

Objective

The objectives of this lesson are for students to explore the patterns of exponential models in tables, graphs, and symbolic forms and to apply what they have learned to make predictions in a real situation.

Overview of the Lesson

In activities that use paper folding and M&M's, students will collect data, create scatterplots, and determine algebraic models that represent their functions. Students begin this lesson by collecting data within their groups. They fold a sheet of paper and determine the area of the smallest region after each fold. Next they draw a scatterplot of their data and determine by hand an algebraic model for it. After determining this algebraic model by hand, students collect exponential decay data using M&M's and use the graphing calculator to determine a model for this data. Both investigations allow students to explore the patterns of exponential models in tables, graphs, and symbolic form. The final activity provides an opportunity for students to apply what they know about exponential models to future populations of African Rhinos.

Materials

- graphing calculator with overhead unit
- overhead projector

For each group of four:
- individual white boards (optional)
- paper
- graph paper
- 2 small bags of M&M's
- *Paper Folding* activity sheets
- *The M&M Investigation* activity sheets
- *African Rhino* activity sheets
- graphing calculators

Procedure

1. **Paper Folding Investigation, Part I**: After the blank paper and activity sheets have been distributed to each pair of students, direct the students through the steps for this investigation, or have groups read and follow the written instructions. One student in each pair is the recorder. The recorder makes a table of values to record the number of folds in column one and the number of sections in column two. The students fold the paper in half and record the number of sections. They should repeat this procedure about five times, each time recording the number of folds and the corresponding number of sections. After the students have made their table of values, a student records this data on the board so that all can verify their work.

   The students determine the rule for the pattern that they see in the table. First they observe the numbers in the table and think about a rule that might represent that pattern. The students then discuss their ideas with the others in their group. After the groups complete this discussion, they report their ideas during a class discussion. These ideas are recorded on the board and any differences that are reported are discussed.

2. **Paper Folding Investigation, Part II**: For the second part of this investigation, the students reverse roles. The new recorder makes a table of values labeling column one as the number of folds and column two as the area of the smallest section. Using a fresh sheet of paper that has an area designated as one unit, the folder makes the appropriate folds and reports the area of the smallest section to the recorder, who completes the table of values.

   Again, the students determine the rule for the pattern that they see in the table. First they observe the numbers in the table, think about a rule that might represent that pattern, and then discuss their ideas with the others in their group. Ideas from the groups are recorded on the board and discussed. It is important for students to understand that $\left(\frac{1}{2}\right)^x$ and $\frac{1}{2^x}$ are equivalent.

   Using graph paper or white boards, students graph the points recorded in the table of values. *Number of Folds* is the independent variable, and *Area of the Smallest Section* is the dependent variable. You may need to suggest an appropriate scale, or have the students discuss what would be an appropriate scale for the vertical axis. After drawing scatterplots, students sketch a smooth curve to approximate the function and discuss the shape of this curve comparing it to the curves for exponential growth models.

3. **The M&M's Investigation**: Students work in pairs and follow instructions for collecting the data. After they shake the bag of M&M's and pour the candies onto the desk, they remove all of the M&M's that do not have the "m" showing. Students then count the remaining candies and record that number in the data table. These candies are then returned to the bag, and this procedure is repeated until there are fewer than 10 candies left. It is important

that students have a number greater than zero as the last entry in the data table.

Students now use a graphing calculator to determine an exponential model for their data. First they enter the data in L1 and L2, and then they create a scatterplot. When they examine the scatterplot, students need to recognize that the shape of the plot indicates that the pattern appears to be exponential. Then they calculate an exponential model using the graphing calculator to graph that model and to see that it fits the scatterplot.

Through class discussion and teacher questioning, students make the connection between the general exponential model, $y = a(b)^x$, their own model, and the M&M's data. Students realize that the value for *a* seems to be close to the number of M&M's they started with, and the value for "b" seems to be approximately $\frac{1}{2}$. They connect this value to the probability that an M&M candy will land with the "m" showing.

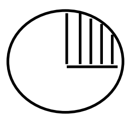
4. **The African Rhino Problem**: This problem allows students to apply what they have learned about exponential models to make predictions about future population figures for African rhinos. Students could work with their groups to complete the problem in class, or the problem could be started in class and completed for homework.

Assessment

It is important for the teacher to informally assess student understanding by observing groups and listening to student responses. Are the students involved? Can they create the data tables and the scatter plots? Do they understand the patterns in the tables, graphs, and algebraic equations? Are they able to apply what they know about exponential models to solve problems? Are they able to operate the graphing calculator successfully? Assessment of student understanding of this lesson will help guide future instruction.

Extensions & Adaptations

- Use *The M&M Investigation* format to explore exponential decay models for other objects. For example, start by tossing 200 objects and using some rule to remove certain objects. Record the number remaining and repeat the procedure. Once the data is collected, use the graphing calculator to determine the actual model.

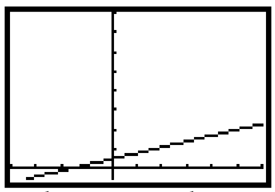
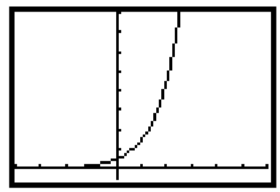
Object	Rule	Theoretical Model
plastic spoons	remove the spoons landing face up	$y = 200(\text{probability of landing face up})^x$
kernels of corn 	spread kernels evenly on a paper plate and remove kernels in the shaded region	$y = 200(0.75)^x$
thumb tacks	remove the tacks landing point up	$y = 200(\text{probability of landing point up})^x$
pennies	remove the tails	$y = 200(0.5)^x$

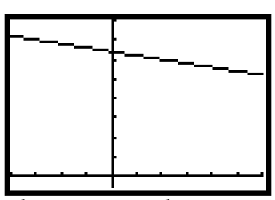
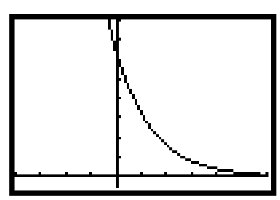
- Assign a project dealing with exponential models. There are many examples of exponential growth and decay that students could investigate. Have students collect data on something that interests them, create a model based on that data, and make a prediction about future figures. Suggested topics might include: number of TV stations, the national debt, number of cases of small pox reported, number of AIDS cases reported, bald eagle population figures, number of one-room school houses, U.S. population, your state or county population, number of CD's sold, number of records sold.

Mathematically Speaking

In today's world, exponential models are an important family of functions that all students need to understand. In the past, these functions were often not introduced until the end of the second year of algebra. Now, however, technology makes exploring exponential models more accessible.

It is important to emphasize the patterns in the tables, graphs, and algebraic rules for the exponential models. Students might find it useful to compare linear models and exponential models as they study families of functions. Consider the simple examples in the following two charts. (Note that on the graphs in these charts the marks on the x-axes indicate 1 unit and the marks on the y-axes indicate 10 units.)

	Linear Model $y = a + bx$	**Exponential Growth Model** $y = a(b)^x$ **for** $b > 1$
rules	$y = 5 + 3x$ There is a constant additive rate.	$y = 5(3^x)$ There is a constant multiplicative rate.
tables	$x =$ 0 1 2 3 4 5 6 7 8 $y =$ 5 8 11 14 17 20 23 26 29 Notice the constant difference. Take any function value and subtract the previous value in the table to determine the constant rate of change, or the slope of the line: 3 for this function.	$x =$ 0 1 2 3 4 5 6 $y =$ 5 15 45 135 405 1215 3645 Notice the common ratio. Take any function value and divide it by the previous value in the table to determine the base of this exponential model: 3 in this case.
graphs	 The graph is a straight line with a slope of 3.	 The graph curves upward at an increasing rate.

	Linear Model $y = a + bx$	**Exponential Decay Model** $y = a(b)^x$ **for** $0 < b < 1$
rules	$y = 64 - 2x$ There is a constant additive rate of negative 2.	$y = 64(0.5)^x$ There is a constant multiplicative rate of one half.
tables	$x =$ 0 1 2 3 4 5 6 7 $y =$ 64 62 60 58 56 54 52 50 Notice the constant difference. Take any function value and subtract the previous value in the table to determine the constant rate of change, or the slope of the line: –2 for this function.	$x =$ 0 1 2 3 4 5 6 7 $y =$ 64 32 16 8 4 1 0.5 0.25 Notice the common ratio. Take any function value and divide it by the previous value in the table to determine the base of this exponential model: 0.5 in this case.
graphs	 The graph is a straight line with a slope of –2.	 The graph curves downward at a decreasing rate.

Tips From Ellen

Structuring Learning Experiences

If it is our goal in the classroom to ensure a high degree of student on task behavior as well as success in carrying out the task, there are some important instructional steps we can make, many of which are illustrated in this lesson. These include:

Build on prior experience. This lesson connects to students' prior learning experiences of general exponential equations as well as specific real life models. It is also clear that students have prior experience in making tables and scatterplots as well as the several calculator tasks required. The mathematical content of the lesson is a logical next step.

Sequence from simple to complex. This lesson provides a transition from exponential growth to decay, with the models moving from simple ratios and integers to more complex decimals.

Move from concrete to abstract. Students use experiential activities with manipulatives to create simple tables. Then they discuss patterns, which are then translated into mathematical models.

Break tasks and instructions into small steps. Students generally have to remember only a few things to do at each point. In many cases, tasks are divided still further so that each student has a part of the task.

Model expectations. The teacher models how to fold the paper, how to set up and label tables, and how to set up and label graphs. It is also clear from the comfortable way in which students carry out routines that these had been well established.

Use and expect precision in language. Clarity is essential in giving and understanding instructions as well as communicating understanding.

Check for understanding before proceeding. Results for each step are checked, often by posting on the board as well as through oral summary, prior to introducing the next step.

Provide necessary resources. In this case, the use of marking pens, large paper, and dry erase boards for graphing make the lesson move smoothly. The use of highly visual recording makes small group sharing as well as teacher monitoring much more efficient.

A caution: by too carefully and consistently structuring tasks, we may eliminate both small and large opportunities for students to engage in authentic problem solving. High school mathematics has historically presented carefully sequenced problems with neat answers. Real life problems are messy and require complex thinking.

Students need opportunities to learn how to break a complex task into small tasks, how to set up their own tables and graphs, how to connect to prior knowledge, and how to identify and obtain necessary resources. They need to share their struggles and thinking with others and learn to identify practices that are effective. These are some of the goals of reform mathematics.

The challenge for all teachers is to determine the line between structuring for short term success in completing a task and structuring for long term success in student risk-taking and complex problem-solving.

Problem-solving is knowing what to do when you don't know what to do.

Resources

Coxford, Art, James Fey, Christian Hirsch, Harold Schoen, Gail Burrill, Eric Hart, and Ann Watkins. *Contemporary Mathematics in Context, A Unified Approach*. Chicago, IL: Everyday Learning, 1997.

National Council of Teachers of Mathematics. *Curriculum and Evaluation Standards for School Mathematics Addenda Series, Grades 5 - 8: Patterns and Functions*. Reston, Virginia: National Council of Teachers of Mathematics, 1991.

The Marker Board People, maker of Graph Board Tablets (1-800-828-3375), 2300 Spikes Lane, Lansing, MI 48906.

Internet location: **http://www.glenbrook.k12.il.us/gbsmat/car/intro.html** This lesson begins with the students choosing a make and model of car and then collecting its blue-book value over several years. The students follow the steps to model the data over an exponential curve. Finally, the students are asked questions based on their curve, and they generate a report from their findings. This is an excellent project.

Internet location: **http://www.cs.rice.edu/~sboone/Lessons/Titles/popclock.html** Students review the census site on the Internet and gather data regarding trends in population. They study this data, make predictions on future populations, and compare their results with information available on the Internet.

Internetlocation: **http://dept.physics.upenn.edu/courses/gladney/mathphys/Contents.html** Students can access various chapters of *The Interactive Textbook of PFP 96*. Topics covered in these interactive lessons include linear, quadratic, and exponential functions, geometry, and basic calculus.

Internet Location: **http://dept.physics.upenn.edu/courses/gladney/mathphys/subsubsection1_1_4_1.html** This address takes you directly to the section on exponential models in The Interactive Textbook mentioned above.

Internet Location: **http://gauss.hawcc.hawaii.edu/maths.html** This site offers a

variety of excellent integrated lessons that connect to a variety of other Internet sites. Examples of topics covered by the lessons include oil resources, population, and bicycle helmets.

Ideas for Online Discussion

(*Some ideas may apply to more than one standard of the* ***NCTM Professional Standards for Teaching Mathematics.)***

Standard 1: Worthwhile Mathematical Tasks

1. Several of the newer mathematics curricula include exponential models as part of their ninth grade program. What advantages do you see for early coverage of this concept?

2. What are some of the ways that teachers can effectively teach calculator skills without sacrificing mathematical content? How do you teacher your students new calculator skills?

Standard 2: The Teacher's Role in Discourse

3. Describe the teachers role in discourse in this lesson. How would you stimulate and moderate discussion when teaching this lesson to your students?

Standard 3: Students' Role in Discourse

4. How many students were actively engaged in discussion during this lesson? What do you do in your classroom to increase student participation both in small group and in total class discussions?

Standard 4: Tools for Enhancing Discourse

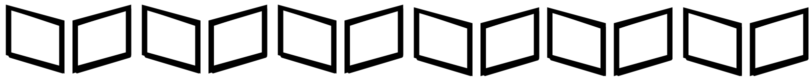
5. In the video lesson, the students do not have activity sheets for the paper folding or for the M&M investigations. What impact did this have on the lesson? There are student activity sheets included with this lesson guide which you may use if you wish. How might the use of the activity sheets change the dynamics of the classroom?

6. Describe the balance between the mathematics that was done without technology and the mathematics done with the aid of the graphing calculator. Why was this appropriate for this particular class and this particular lesson?

7. What tools were used to enhance the group work? What tools do you use in your classroom to enhance group work?

Standard 6: Analysis of Teaching and Learning

8. What evidence is there that the students in the video class understand the lesson? What would you do in your own classroom to assess student understanding of this lesson?

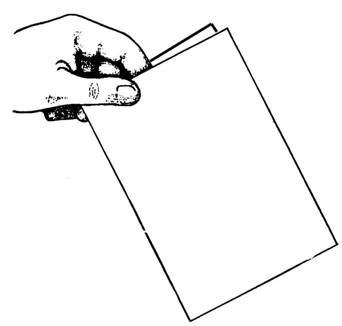


The Paper Folding Activity

Part I: Number of Sections

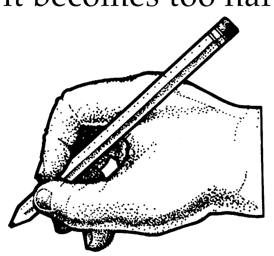
Number of Sections

Number of Folds	Number of Sections
0	
1	
2	
3	
4	
5	
6	

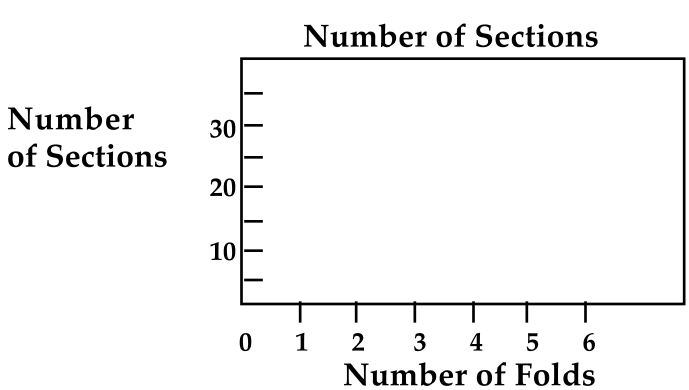


1. Fold an 8.5 x 11″ sheet of paper in half and determine the number of sections the paper has after you have made the fold.

2. Record this data in the table and continue in the same manner until it becomes too hard to fold the paper.



3. Make a scatter plot of your data.



4. Determine a mathematical model that represents this data by examining the patterns in the table.

5. What might be different if you tried this experiment with an 8.5 x 11″ sheet of wax paper or tissue paper?

Part II: Area of Smallest Section

Area of Smallest Section

Number of Folds	Area of Smallest Section
0	1
1	
2	
3	
4	
5	
6	

6. Fold an 8.5 x 11″ sheet of paper in half and determine the area of the smallest section after you have made the fold.

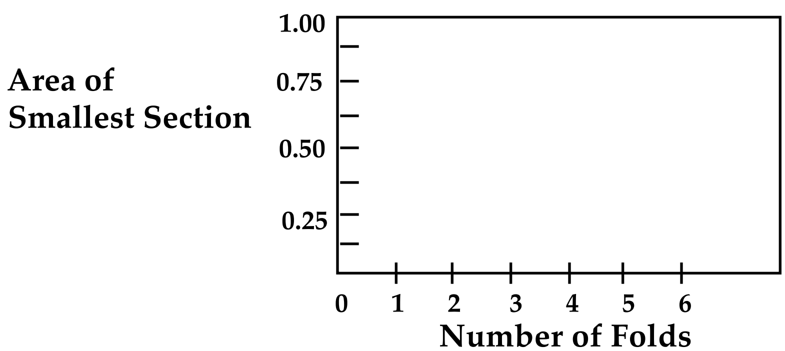
7. Record this data in the table and continue in the same manner until it becomes too hard to fold the paper.

8. Make a scatter plot of your data.

9. Determine a mathematical model that represents this data by examining the patterns in the table.

Did you know that paper was invented by the Chinese about 105 AD?

Area of Smallest Section



The "M&M"® Investigation

Part 1: Collecting Data

1. Empty your bag of M&M's and count them. Then place them back in the bag, and mix them well. Pour them out on the desk, count the number that show an "m", and place these back in the bag. The others may be eaten or removed. Record the number that show an "m" in your data table then repeat this procedure. Continue until the number of M&M's remaining is less than 5 but greater than 0.

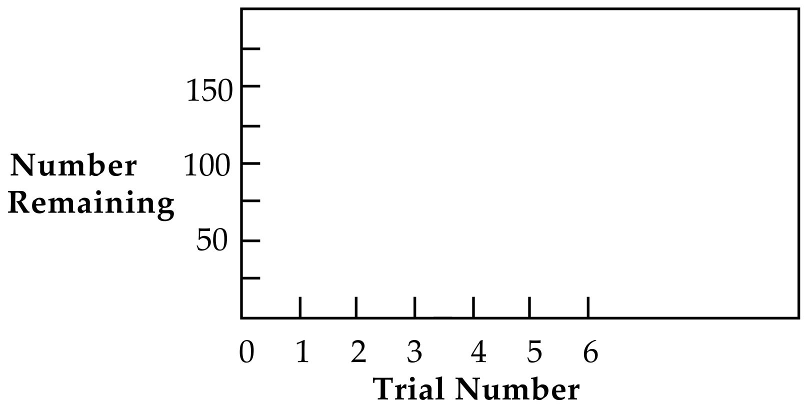
Number of M&M's Remaining

Trial Number	Number of M&M's remaining
0	
1	
2	
3	
4	
5	
6	

Part 2: Graphing and Determining the Exponential Model

2. Use a graphing calculator to make a scatterplot of your data. Copy your scatter plot onto the grid below. The use the graphing calculator to determine an exponential model, and graph that equation. Sketch in the graph, and write your exponential equation.

Number of M&M's Remaining



Equation: _________

Part 3: Interpreting the Data

3. In your model, $y = a(b)^x$, what value do you have for a? What does that seem to relate to when you consider your data? When $x = 0$, what is your function value? Compare this to the values in your data table.

4. What is the value for b in your exponential model? How does this value relate to the data collection process?

5. How does the M&M experiment compare to the paper folding activity? How are they alike and how are they different?

Directions For Using the TI-82 to: Graph a Scatter Plot, Determine an Exponential Model, and Graph the Model

To enter data:

STAT 1:EDIT ENTER

(Clear all lists by using the up arrow button to go to the top and pushing clear.)

Enter data with L1 as the independent variable and L2 as the dependent variable.

To plot scatter graph:

2nd Y (STAT PLOT) 1:PLOT1 ENTER

Plot on; type scatter (1st).

X list: L1; Y list: L2; Mark: 1st symbol.

To choose window and graph:

ZOOM 9:ZoomStat ENTER

YOU SHOULD SEE A SCATTER PLOT.

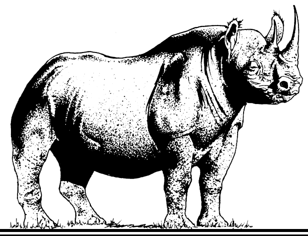
To find the exponential regression:

STAT CALC A:ExpReg ENTER

To graph the line:

Y= VARS 5:Statistics EQ 7:RegEQ GRAPH

YOU SHOULD GET A CURVE THOUGH THE SCATTER PLOT.



The African Black Rhino Population

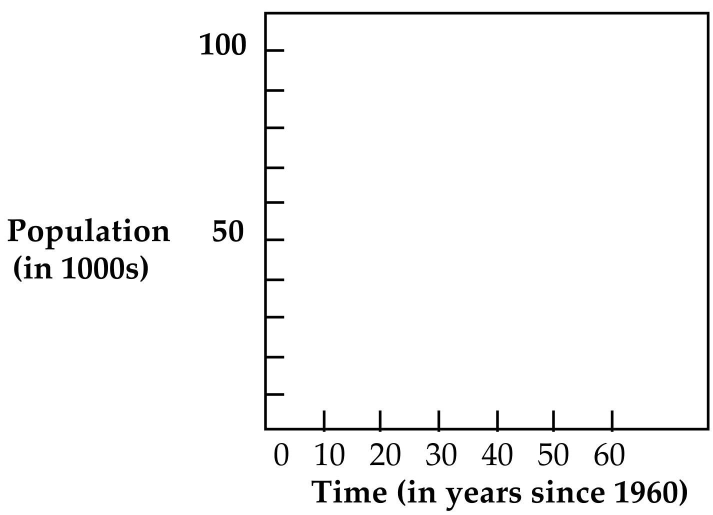
An Application of Exponential Models

African Black Rhino Population

Year	Population (in 1000s)
1960	100
1980	15
1991	3.5
1992	2.4

1. Make a scatterplot for the given data.

African Rhino Population



2. Use your graphing calculator to create an algebraic model.

3. Predict the black rhino population for the years 1998 and 2004.

4. Use your model to determine the rhino population in 1950.

5. Should scientists be concerned about this decrease in population?

6. Compare your equation for your M&M data to your equation for the rhino data. How are they alike, and how are they different?

The Paper Folding Activity

Selected Answers

Part I: Number of Sections

Number of Sections

Number of Folds	Number of Sections
0	1
1	2
2	4
3	8
4	16
5	32
6	64

4. $y = 2^x$

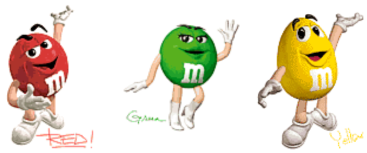
5. The results would be exactly the same, but you would be able to make more folds and collect more data because the paper would be thinner.

Part II: Area of Smallest Section

Area of Smallest Section

Number of Folds	Area of Smallest Section
0	1
1	1/2
2	1/4
3	1/8
4	1/16
5	1/32
6	1/64

9. $y = (1/2)^x$



The "M&M"® Investigation

Selected Answers

1. Answers will vary.

Number of M&M's Remaining

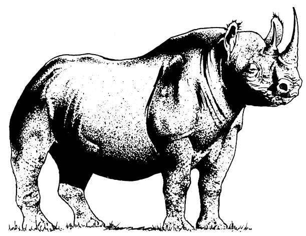
Trial Number	Number of M&M's remaining
0	140
1	76
2	39
3	22
4	12
5	8
6	3

2. $y = 140.1(0.54)^x$

3. a = 140.1. This is almost the original number of candies. When $x = 0$, the value of the function is 140.1, or a which is very close to the original data.

4. The value for b is 0.54, which is close to the probability that an M&M candy will land with the "m" showing. As the data is collected the number of M&M's decreases by almost half each time.

5. The M&M data pattern is very close to the data pattern for the paper folding activity when we were recording the trial number and the area of the smallest section. The starting numbers are different, but the pattern of decrease is almost the same—each term is about half of the previous term. In the case of the paper, it is exactly half; in the case of the candy, it is approximately half.



The African Black Rhino Population

Selected Answers

An Application of Exponential Models

African Black Rhino Population

Year	Population (in 1000s)
1960	100
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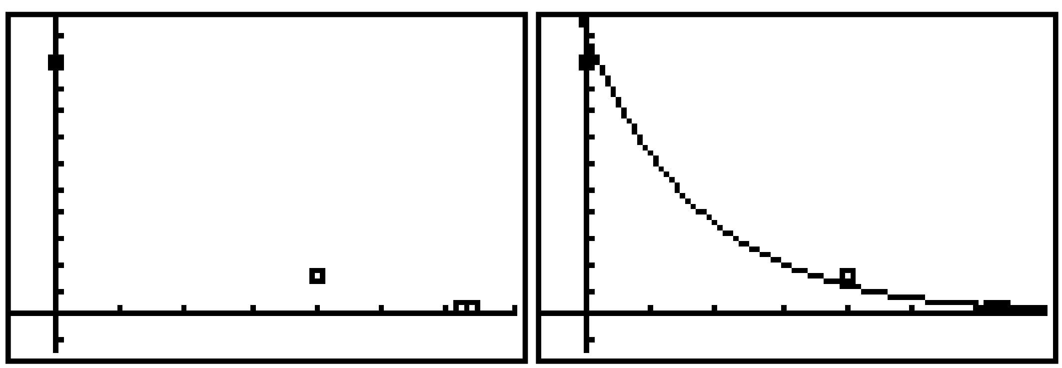
1.



2. $y = 110538(0.893)^x$

3. 1998 (year 38) 1,499
   2004 (year 42) 953

4. 1950 (year –10) 342,765

5. Yes, the population is decreasing rapidly! In about 100 years, the rhinos will be extinct.

6. The a values are different, and the b values are different, but both models represent exponential decay.