Illustrative Mathematics

S-CP Alex, Mel, and Chelsea Play a Game

Alignment 1: S-CP.B.9

Not yet tagged

Alex, Mel, and Chelsea play a game that has 6 rounds. In each round there is a single winner, and the outcomes of the rounds are independent. For each round the probability that Alex wins is $\frac{1}{2}$, and Mel is twice as likely to win as Chelsea. What is the probability that Alex wins three rounds, Mel wins two rounds, and Chelsea wins one round?

Commentary

This task combines the concept of independent events with computational tools for counting combinations, requiring fluent understanding of probability in a series of independent events. This unscaffolded problem requires students to complete several steps, and realize the need for such steps, illustrating practice standard MP1: Make sense of problems and persevere in solving them.

After setting up the problem, students will need to know that probabilities for independent events are multiplied and then take into account all of the different scenarios which result in the desired outcome (Alex winning 3 rounds, Mel 2, and Chelsea 1). The solution provided uses the binomial coefficient notation for the number of choices, but this is not required.

This task was adapted from problem #11 on the 2012 American Mathematics Competition (AMC) 12A Test.

For the 2012 AMC 12A, which was taken by 72,238 students, the multiple choice answers for the problem had the following distribution:

Choice	Answer	Percentage of Answers
(A)	$\frac{5}{72}$	9.35
(B)*	$\frac{5}{36}$	20.51
(C)	$\frac{1}{6}$	12.83
(D)	$\frac{1}{3}$	6.16
(E)	$1.$	19.48
Omit	--	31.63

Of the 72,238 students: 28,268, or 39%, were in 12th grade; 34,124 or 47%, were in 11th grade; 4,615, or 6%, were in 10th grade; and the remainder were below 10th grade.

Solution: 1

There are three separate steps to this problem. We need to calculate the probabilities that Mel and Chelsea win a single round. Next, we must compute the probability of any given scenario where Alex wins 3 rounds, Mel wins 2, and Chelsea wins 1 round. Then we have to find out how many ways there are for Alex to win three rounds, Mel 2, and Chelsea 1. With all of this information we can then calculate the probability that Alex wins 3 rounds, Mel wins 2, and Chelsea wins 1.

We are given that Mel is twice as likely to win a round as Chelsea. Their combined probability of winning a round is $\frac{1}{2}$ since this number, combined with Alex's probability of winning a round, must add up to 1. If we let x denote Chelsea's probability of winning a given round then $2x$ is Mel's probability of winning a round and

$$2x + x = \frac{1}{2}$$

so $x = \frac{1}{6}$. So the probability that Mel wins a round is $\frac{2}{6} = \frac{1}{3}$and the probability that Chelsea wins a round is $\frac{1}{6}$.

Suppose we let AAAMMC denote the situation where Alex wins the first three rounds, Mel wins the fourth and fifth, and Chelsea wins the sixth. To find the likelihood that this happens, we use the information from the previous paragraph about the probability that each individual wins a single round. So the probability that Alex wins the first round is $\frac{1}{2}$. Similarly the probability that Alex wins round 2 is $\frac{1}{2}$ as is the probability that he wins round 3. The probability that Mel wins round 4 is $\frac{1}{3}$ and so is the probability that Mel wins round 5. Finally the probability that Chelsea wins round 6 is $\frac{1}{6}$. Since the different rounds are independent events, this means that the probability of an outcome such as AAAMMC (with three wins for Alex, two wins for Mel and one win for Chelsea) is

$$\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{3}\right)^2 \times \left(\frac{1}{6}\right)$$

This is the same as the probability of, say, MMAAAC, where Mel wins the first two rounds, Alex the next three, and Chelsea the final one. This brings us to our last step, finding out how many ways there are for Alex to win 3 rounds, Mel 2 and Chelsea 1.

Since there are six rounds, the number of ways Alex can win 3 of them is the combinatorial symbol

$$\binom{6}{3}$$

which is equal to 20. Once the three games which Alex has won have been chosen, Mel must win two of the other three and there are

$$\binom{3}{2}$$

for this to happen: this is equal to 3. The remaining game, won by Chelsea, has no further choices.
Putting together all of this information, the probability that Alex wins three games, Mel two, and Chelsea one is

$$\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{3}\right)^2 \times \left(\frac{1}{6}\right) \times 20 \times 3 = \frac{5}{36}.$$